TABLE 10-4 *Representation of df Associated with the SS_A, SS_B, and $SS_{A\times B}$*

	Factor A				
Factor B	a_1	a_2	a_3	a_4	Sum
b_1				X	B_1
b_2				X	B_2
b_3	X	X	X	X	X
Sum	A_1	A_2	A_3	X	T

What about the AB_{ij} sums within the body of the matrix? For any one of the columns, the sum of the cell totals must equal the corresponding marginal total. This places one restriction on each of the columns; these restrictions are represented by **X**'s in the row at level b_3. (This row was again picked arbitrarily.) A similar restriction is placed on the rows: the sum of the cell totals in any one of the rows must equal the corresponding marginal totals. These restrictions are indicated by **X**'s in the column at level a_4. The unmarked cells, then, represent the *df* for the $A \times B$ interaction. This rectangle is bounded on one side by $a - 1$ columns and on the other side by $b - 1$ rows, and the total number of "free" cells without **X**'s is the quantity $(a - 1)(b - 1)$.

The *df* for the within-groups sum of squares ($SS_{S/AB}$) follow the same general rule for the determination of the number of *df*. Since this sum of squares consists of a within-group sum of squares that is pooled over the *ab* groups, we can start by finding the *df* for each individual cell in the matrix:

$$df_{S/AB_{ij}} = s - 1.$$

One *df* is lost because of the restriction that the *s* different *ABS* scores must sum to the cell total (AB_{ij}). Now, if we sum the *df*'s over the *ab* cells, we obtain

$$df_{S/AB} = \sum df_{S/AB_{ij}} = ab(s - 1).$$

Mean Squares and F Ratios

The mean squares are found by dividing each sum of squares by its corresponding *df*:

$$MS = \frac{SS}{df}.$$

These are enumerated in the fourth column of Table 10-3. The F ratios are formed in each case by dividing the mean squares by the $MS_{S/AB}$:

$$F_A = \frac{MS_A}{MS_{S/AB}}, \quad F_B = \frac{MS_B}{MS_{S/AB}}, \quad F_{A\times B} = \frac{MS_{A\times B}}{MS_{S/AB}}.$$

These F ratios are evaluated in the F table under the appropriate numerator ar denominator *df*'s.

The logic behind the construction of these ratios is the same as that offer in the single-factor case. Briefly, each mean square in the numerator of the ratio is assumed to provide a population estimate of the particular effect pl error variance. The denominator in the F ratio ($MS_{S/AB}$) is assumed to provi an estimate of error variance alone. The null hypothesis in each case is that t population treatment effects (the effects due to factor A, to factor B, and to t interaction of the two factors) are zero. The within-groups mean squa ($MS_{S/AB}$) is the appropriate error term for the three mean squares because t numerator mean squares (MS_A, MS_B, and $MS_{A\times B}$) and the denominator me square ($MS_{S/AB}$) are independent estimates of error variance when the n hypothesis is true. Under these circumstances, the expected values of the th F ratios will be approximately 1.0. We follow the same decision rules as outlir for the single-factor case (pp. 63–65). A significant F indicates that the n hypothesis is untenable, and we then accept the alternative hypothesis that th are treatment effects present.

Orthogonality of the Two-Way Analysis

It is possible to show that the SS_A, SS_B, and $SS_{A\times B}$ are mutually orthogona i.e., that they provide independent information about the outcome of the exp ment. (See Appendix A for a proof of this statement.) As we noted in Chapte these tests are not *statistically* independent, since the same error term is usec form each F ratio. This, however, does not appear to present a problem for interpretation of the results of an experiment. The three sources of varia extracted in the analysis we have been discussing represent an efficient wa dividing up the *df* associated with the total between-groups sum of squares would certainly be possible to divide this sum of squares into a *different* se orthogonal comparisons. Nevertheless, investigators present most facto experiments in a way that makes obvious their intention to extract anc evaluate the sources of variance listed in Table 10-3. In actuality, then, we think of the analysis of a two-way factorial experiment as consisting of a se *planned orthogonal comparisons*. But we are not restricted to these c parisons alone. Often we will want to isolate the locus of a significant r effect or of an interaction. Procedures for accomplishing these compari will be discussed in subsequent chapters of Part III.

ASSUMPTIONS UNDERLYING THE ANALYSIS

Except for the basic structural model, the assumptions for the two-fa experiment are the same as those we enumerated for the single-factor First, the model under which we have been operating merely specifie components of variance with which we are already familiar.[3] Briefly,

$$ABS_{ijk} = \mu + (\mu_i - \mu) + (\mu_j - \mu) + (\mu_{ij} - \mu_i - \mu_j + \mu) + (ABS_{ijk} - \mu_{ij}).$$

That is, ABS_{ijk} can be represented by the sum of the following quantities: (1) the overall mean of the population (μ), (2) the treatment effect at level a_i $(\mu_i - \mu)$, (3) the treatment effect at level b_j $(\mu_j - \mu)$, (4) the interaction effect at cell ab_{ij} $(\mu_{ij} - \mu_i - \mu_j + \mu)$, and (5) experimental error $(ABS_{ijk} - \mu_{ij})$.

In order for the F ratios to be distributed as F, three additional assumptions must be met: (1) each of the ab treatment populations is normally distributed, (2) the variances of the treatment populations are equal, and (3) the error components $(ABS_{ijk} - \mu_{ij})$ are independent within groups and between treatment groups as well.

As we noted in the previous discussion of assumptions underlying the single-factor analysis of variance (see pp. 74–77), the assumptions that the within-group variances are normally distributed and homogeneous are not critical when the sample sizes are of a reasonable size (greater than 10, say) and are equal. The same conclusion holds for the two-factor experiment. In short, we can safely disregard all except major violations of these two assumptions. The assumption of independence of the individual error components, of course, continues to be important. As we saw in Chapter 5, independence is achieved by randomly assigning subjects to the different treatment conditions. If each subject is equally likely to be given any one of the ab treatments, then the ABS scores and hence, $ABS_{ijk} - \mu_{ij}$, will be independent.

NUMERICAL EXAMPLE

We are now ready for a numerical example showing all of the steps required for the analysis of a two-way factorial experiment. The example consists of a hypothetical investigation of the role of drive level and magnitude of reward on the learning of a discrimination problem by monkeys. The animals are given five trials a day for four days on a set of 20 "oddity" problems. In this task, three objects (two the same, one different) are presented to the monkeys, and the subject's task is to learn to select the nonduplicated (odd) object. A food reward is placed in a well underneath the correct object. A trial consists of the presentation of the three objects and the monkey's selection of one of them. The response measure is the number of correct selections in the 20 training trials. One of the independent variables (factor A) is the magnitude of the food reward, either 1, 3, or 5 grapes, while the other variable (factor B) is the drive level of the animals, either 1 hour of food deprivation or 24 hours of food deprivation. Four monkeys are randomly assigned to each treatment combination. Thus, the design is a 3×2 factorial with $s = 4$ subjects. The individual ABS scores appear in the upper portion of Table 10-5.

[3] For a more complete description of the statistical model, see Chapter 16.

The individual cell sums (AB_{ij}) are entered into an AB matrix in the middle portion of the table. Each cell total is obtained by summing the four observations in each cell. For cell ab_{11},

$$AB_{11} = \sum ABS_{11k} = 1 + 4 + 0 + 7 = 12.$$

After obtaining these cell sums, it is usually a good idea to plot the cell means so that we can see more readily whether or not we have obtained an interaction. This has been done in Fig. 10-1 (the means are presented in the bottom portion of the table). The figure shows a rather sizable interaction of the two variables. It appears that magnitude (factor A) has little differential effect upon correct responding by hungry monkeys and an increasing positive effect with less hungry monkeys. In other words, hungry animals are unaffected by the differences in the size of the food reward, while less hungry animals are. Now that we have a "feel" for the way the experiment came out (and what the analysis should reveal), we can proceed with the calculations.

TABLE 10-5 *Numerical Example: Two-Factor Analysis*

ABS MATRIX (INDIVIDUAL OBSERVATIONS)

Treatment Combinations

ab_{11}	ab_{12}	ab_{21}	ab_{22}	ab_{31}	ab_{32}
1	15	13	6	9	14
4	6	5	18	16	7
0	10	7	9	18	6
7	13	15	15	13	13

AB MATRIX (SUMS)

Drive (Factor B)	Amount of Food (Factor A): 1 Grape (a_1)	3 Grapes (a_2)	5 Grapes (a_3)	Sum
1 hour (b_1)	12	40	56	108
24 hours (b_2)	44	48	40	132
Sum	56	88	96	240

AB MATRIX (MEANS)

Factor B	Factor A: a_1	a_2	a_3
b_1	3.0	10.0	14.0
b_2	11.0	12.0	10.0

The first step is to substitute these data into the computational formulas for the sums of squares given in Table 10-3. We will perform these operations in two steps: (1) the calculation of the basic terms entering into the computational formulas and (2) the addition and subtraction of these terms in the actual determination of the various sums of squares. We will now solve for these basic quantities and identify each by means of the letter code. Working first with the AB matrix, we obtain

$$[T] = \frac{(T)^2}{abs} = \frac{(240)^2}{3(2)(4)} = \frac{57{,}600}{24} = 2400.00,$$

$$[A] = \frac{\sum (A)^2}{bs} = \frac{(56)^2 + (88)^2 + (96)^2}{2(4)} = \frac{20{,}096}{8} = 2512.00,$$

$$[B] = \frac{\sum (B)^2}{as} = \frac{(108)^2 + (132)^2}{3(4)} = \frac{29{,}088}{12} = 2424.00,$$

and

$$[AB] = \frac{\sum (AB)^2}{s} = \frac{(12)^2 + (44)^2 + \cdots + (56)^2 + (40)^2}{4}$$

$$= \frac{10{,}720}{4} = 2680.00.$$

The final calculation requires the scores presented in the ABS matrix:

$$[ABS] = \sum (ABS)^2 = (1)^2 + (4)^2 + \cdots + (6)^2 + (13)^2 = 3010.$$

We can now obtain the sums of squares by combining the quantities we have just calculated in the patterns specified in Tables 10-2 or 10-3. This has been done twice in Table 10-6. In the upper portion of the table we have indicated the actual numbers entering into the determination of each sum of squares,

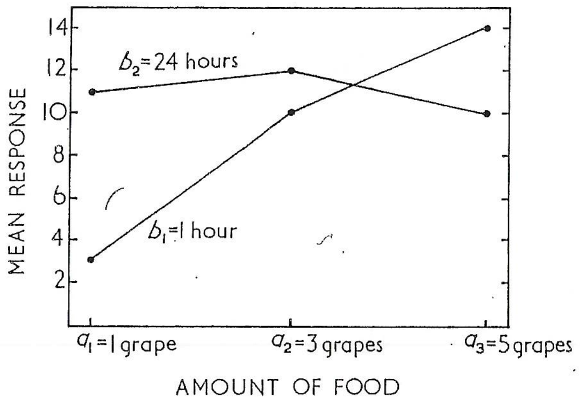

Fig. 10-1 Plot of data presented in Table 10-5.

TABLE 10-6 *Summary of the Analysis*

UNCODED

Source	Calculations	SS	df	MS	F
A	2512.00 − 2400.00 =	112.00	2	56.00	3.06
B	2424.00 − 2400.00 =	24.00	1	24.00	1.31
$A \times B$	2680.00 − 2512.00 − 2424.00 + 2400.00 =	144.00	2	72.00	3.93*
S/AB	3010 − 2680.00 =	330.00	18	18.33	
Total	3010 − 2400.00 =	610.00	23		

* $p < .05$.

CALCULATIONS CODED BY LETTERS IN BRACKETS

Source	Calculations[a]	SS	df	MS	F
A	2512.00 − 2400.00 =	112.00	2	56.00	3.06
B	2424.00 − [T] =	24.00	1	24.00	1.31
$A \times B$	2680.00 − [A] − [B] + [T] =	144.00	2	72.00	3.93*
S/AB	3010 − [AB] =	330.00	18	18.33	
Total	[ABS] − [T] =	610.00	23		

* $p < .05$.
[a] Bracketed letters represent complete terms in the computational formulas; a particular term is identifie by the letter(s) appearing in the numerator.

while in the lower portion we have represented the same operations by replacin calculations that recur in subsequent rows of the table with the coded quantitie This latter representation is the form that the summary tables will usuall take in the remainder of the book. The coding reduces the amount of writin needed to capture the critical steps in the analysis, and it emphasizes the patter in which sums of squares are calculated.

The final steps of the analysis are recorded on the right sides of both tables i an analysis-of-variance summary table. We should check for computation errors in our calculations by summing the component sums of squares to veri that the total equals the SS_T. The df's are found by a simple substitution the corresponding formulas:

$$df_A = a - 1 = 3 - 1 = 2,$$

$$df_B = b - 1 = 2 - 1 = 1,$$

$$df_{A \times B} = (a - 1)(b - 1) = (3 - 1)(2 - 1) = 2(1) = 2,$$

$$df_{S/AB} = ab(s - 1) = 3(2)(4 - 1) = 3(2)(3) = 18,$$

$$df_T = abs - 1 = 3(2)(4) - 1 = 24 - 1 = 23.$$

The mean squares are obtained by dividing each sum of squares by t appropriate df. Finally, each mean square reflecting the contribution ol different component of interest is tested against the mean square representi experimental error—i.e., the within-groups mean square ($MS_{S/AB}$). The observed F ratios are compared with the critical values of F listed in Table C-1. In this example, only the interaction source of variance reaches an acceptable level of significance.

Interpretation of Main Effects

In this example, clearly, we would be foolish to conclude that magnitude of reward and food deprivation were ineffective variables merely on the basis of the nonsignificant main effects. On the contrary, both independent variables influence learning, but their influence is masked by the presence of an $A \times B$ interaction. A significant interaction tells us that the variables do affect behavior, but only in combination with particular levels of another variable. In this experiment, the amount of reward does not produce differences in learning when the monkeys are hungry ($b_2 = 24$ hours), but it does produce an impressive effect when the animals are only slightly hungry ($b_1 = 1$ hour). We can restate this interaction in terms of the other independent variable. That is, drive is relatively ineffective when the animals are given 3 or 5 grapes (a_2 and a_3, respectively), but hungry animals show learning superior to that of less hungry animals when the reward is small ($a_1 = 1$ grape).

In general, the interpretation of any main effect depends upon the presence or absence of significant interaction effects. If there is no interaction, the outcome of the F tests involving the main effects can be interpreted without qualification. With an interaction, on the other hand, the meaning of these F tests must be interpreted with caution. We have already seen that the presence of an interaction and nonsignificant main effects does not mean that the independent variables are ineffective.

But what if the interaction is significant *and* we find one or even two significant main effects? The interpretation of these main effects will also be tempered by a consideration of the interaction; nevertheless, there might still be interest in the *main effects* as such. Consider the example presented in Fig. 10-2. On the left, the two curves do not cross within the limits of the factor A selected for the experiment. In spite of the interaction showing that the largest difference between b_1 and b_2 is found at a_2, it is also true that b_1 is *consistently* above b_2. This is an example of an *ordinal interaction*, where the relative ranking of the levels of factor B in this case does *not* change at the different levels of factor A. In this situation, it would be appropriate to conclude that in general the treatment represented by level b_1 results in performance that is higher than the treatment at level b_2. If we replot (on the right), with factor B on the baseline, we see what is called a *disordinal interaction*. In this view of the experiment, the rank order of the levels of factor A *changes* at the different levels of factor B. No general conclusion may be reached concerning the influence of factor A.

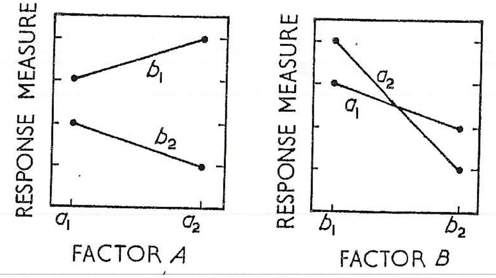

Fig. 10-2 Example of an ordinal interaction (left panel) and of a disordinal interaction (right panel).

The example shows that the ordinality of an interaction depends upon how the results are plotted. Thus, before significant main effects are interpreted and if there is an interaction present, it is wise to plot the data both ways (or to look at the values within the AB matrix both with regard to the rows and to the columns) to see whether or not ordinality exists. If it does, the main effect may be interpreted as a main effect. If it does not, the main effect cannot be interpreted independently of the interaction.