8. In this experiment $a = 5$ levels that are equally spaced on some stimulus dimension. We will assume $s = 8$ subjects in each treatment condition and that $\sum (AS)^2 = 1285$. Suppose we obtained the following sums:

a_1	a_2	a_3	a_4	a_5
34	60	47	20	48

(a) Perform a one-way analysis of variance with these data.
(b) Conduct a trend analysis using the coefficients of the orthogonal polynomial.

9. Suppose we have an experiment with independent groups of $s = 7$ subjects randomly assigned to each of 8 treatment conditions. The error term, $MS_{S/A} = 58.65$. The treatment sums are given below:

a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8
316	333	307	373	398	227	123	436

(a) Is the omnibus F significant?
(b) Conduct multiple comparisons on all comparisons between pairs of groups with the Newman-Keuls and Tukey tests, using $\alpha = .05$. Summarize these tests two ways: (1) by means of a table (see Table 8-3, p. 141) and (2) by means of a listing of overlapping conditions (see Table 8-4, p. 143).

10. Assume that we have a control group and seven experimental groups, with $s = 16$ subjects for each group. The $MS_{S/A} = 28.75$. The totals for each group are given below:

C	E_1	E_2	E_3	E_4	E_5	E_6	E_7
289	270	241	279	191	213	205	198

(a) Is the overall F significant?
(b) Use Dunnett's test to determine which of the treatment means is significantly different from the mean of the control group. Use a two-tailed test at $\alpha = .05$.
(c) Make the same set of comparisons with the Scheffé procedure, $\alpha = .05$. Do your conclusions change?

PART III

Factorial Experiments With Two Factors

In this part we will consider experiments where treatment conditions are classified with respect to the levels represented on *two* independent variables. In Part IV we will go on to discuss experiments involving the concomitant manipulation of three or more independent variables. In all of these discussions we will be assuming that subjects serve in only one of the treatment conditions, that they provide only a single score or observation, and that they are randomly assigned to one of the conditions. Formally, we refer to these sorts of experiments as *completely randomized designs.*[1]

Numerical examples illustrating arithmetic operations discussed in this section may be found at the end of Part III (pp. 247–250).

[1] We will be assuming the so-called *fixed-effects model,* which is appropriate for most of the research in the behavioral sciences. This and other models are discussed in Chapter 16.

chapter nine

THE ADVANTAGES OF FACTORIAL DESIGN AND ITS UNIQUE CONTRIBUTION: INTERACTION

The most common means by which two or more independent variables are manipulated in an experiment is a *factorial arrangement of the treatments* or, more simply, a *factorial experiment* or *design.* We will use these terms interchangeably. In a factorial design, the experiment includes every possible combination of the levels of the independent variables. Suppose, for example, that two variables are manipulated concurrently in a study—the magnitude of the food reward given to a hungry rat for completing a run through a maze and the difficulty of the maze he will be given to learn. We will assume there are three levels of food magnitude (small, medium, and large) and two levels of maze difficulty (easy and hard). The factorial arrangement of the treatment conditions is specified by the six cells in Table 9-1. We will often call such an arrangement a factorial *matrix* or simply a matrix. The cells in the matrix represent the following treatment combinations: small-easy, small-hard, medium-easy, medium-hard, large-easy, and large-hard. Each magnitude of reward (represented by the columns) is combined with each type of maze (represented by the rows). Factorial designs are sometimes referred to as experiments in which the independent variables are completely *crossed.* We can think of the crossing in terms of a *multiplication* of the levels of the different independent variables. In the present example, the treatment combinations may be enumerated by multiplying (small + medium + large) by (easy + hard) to produce the six treatment combinations of the design.

TABLE 9-1 *An Example of a Two-Variable Factorial Experiment*

	Reward Magnitude		
Type of Maze	Small	Medium	Large
Easy			
Hard			

ADVANTAGES OF THE FACTORIAL EXPERIMENT

A great deal of the research in the behavioral sciences consists of the identification of variables contributing to a given phenomenon. Quite typically, an experiment may be designed to focus attention upon a single factor. If the experimenter thinks a factor is important, he may attempt to establish the functional relationship between the independent and dependent variables by including a number of levels of the variable in a single-factor experiment. A main characteristic of this type of investigation is that it represents an assessment of how a variable operates under "ideal" conditions—with all other important variables held constant across the different conditions. An alternative approach is to study the influence of one independent variable in conjunction with variations in one or more additional independent variables. Here the primary question is whether or not a particular variable studied concurrently with other variables will show the same effect as it would when studied in isolation.

Both types of experiments certainly have their place in the behavioral sciences. The manipulation of a single variable in an experiment is most useful when its combination with other independent variables is relatively simple. When the combination is complex, the results of single-factor experiments will give an inaccurate picture of the effect of the variable under study.

The factorial experiment is probably most effective at the *reconstructive* stage of a science, where investigators begin to approximate the "real" world by manipulating a number of independent variables simultaneously. Of course, the type of experiment chosen by a researcher depends upon the complexity with which the phenomenon under study is determined. But it is clear that the factorial experiment has advantages of economy, control, and generality.

Economy

Suppose we are putting together a reading series for use in elementary schools and that we have reason to believe that the format of the books will influence reading speed. Two independent variables that might be of interest are the length of the printed lines and the contrast between the printed letters and the paper. Assume that we choose three line lengths (3, 5, and 7 inches) and three different levels of contrast (low, medium, and high). If we were to manipulate the variables in two separate single-factor experiments, the designs might look like those presented in the upper part of Table 9-2. In the experiment on the left, there is a total of 45 subjects (*S*s), with $s = 15$ subjects assigned to each of the three length conditions. In the experiment on the right, the same number of subjects ($s = 15$) would be randomly assigned to each of the three levels of contrast. Other than differences in line length, on the one hand, and print-paper contrast, on the other, all of the subjects would be treated alike. At the completion of the two experiments, we would be able to analyze the data with the techniques discussed in Part II and make statements concerning the influence of line length and contrast on speed of reading.

TABLE 9-2 *Comparison of One- and Two-Factor Designs*

SEPARATE SINGLE-FACTOR EXPERIMENTS

Line Length (Inches)			Print-Paper Contrast		
3	5	7	Low	Medium	High
15 *S*s[a]	15 *S*s	15 *S*s	15 *S*s	15 *S*s	15 *S*s

FACTORIAL ARRANGEMENT

Print-Paper Contrast	Line Length (Inches)		
	3	5	7
Low	5 *S*s	5 *S*s	5 *S*s
Medium	5 *S*s	5 *S*s	5 *S*s
High	5 *S*s	5 *S*s	5 *S*s

[a] *S* = subject.

Compare these two single-factor experiments with the factorial design presented in the bottom half of Table 9-2, in which the same two variables are manipulated simultaneously. In this experiment the two independent variables are completely crossed, meaning that all possible combinations of the three levels of the two variables are represented. Since each variable has three levels in this example, there is a total of $3 \times 3 = 9$ unique treatment groups. This design would be called a 3×3 factorial (read "three by three"). It should be noted that the sample size in each of the groups is $s = 5$. This number was chosen to provide a comparison with the two single-factor experiments. That is, we start this experiment by obtaining 45 school children; we then randomly assign 5 subjects to serve in each of the 9 treatment combinations.

After the experiment is completed, we will have 5 reading scores in each cell of the matrix. What if we want to obtain an estimate of the average effects of line length on reading speed? This information is obtained easily enough by collapsing across the levels of the other variable (contrast) and dividing by the number of scores ($s = 15$). That is, the mean for the 3-inch condition is found by summing the 15 reading scores in the first column of the matrix (5 scores each from the low-, medium-, and high-contrast conditions) and dividing by $s = 15$. The average performance of the subjects receiving 5- and 7-inch lines is obtained in a similar fashion. Turn now to a determination of the average effects of the other independent variable. The average effects of the low-, medium-, and high-contrast conditions are calculated by collapsing across the length classification—i.e., adding together the scores from the three levels of line length for each of the contrast conditions, and dividing by $s = 15$.

These average estimates of the influence of line length and of contrast are based upon the *same* number of subjects (15) as were the estimates provided by the two single-factor experiments. But note, the factorial experiment produces these estimates much more economically, with only half the number of subjects. The economy of the factorial design represents a distinct advantage over separate single-factor studies.

Experimental Control

In the preceding example, both of the independent variables were of scientific interest to us. That is, we were interested in the influence of each of the variables on reading speed. (This was implied when we considered conducting two single-factor experiments.) There will be times when we turn to a factorial experiment, not so much to obtain information on the two variables, but as a way of controlling important but unwanted sources of variability. (We will discuss this use of the factorial design more thoroughly in Chapter 15.)

The most common example of the use of a factorial experiment to control variability is with *subject variables.* Suppose we wanted to study the length variable in a single-factor experiment, but we knew that differences in the intelligence of the subjects would contribute to an especially large within-groups mean square. Under these circumstances, we would need a fairly strong between-groups effect to produce a significant F ratio. One way to solve this problem would be to select a group of subjects who are relatively homogeneous in intelligence (e.g., restrict IQ to the range 100–110) and to assign them at random to the three length conditions. The within-groups mean square will be smaller in this case, since the variability of subjects treated alike will be smaller with the restricted groups of subjects than with the unrestricted ones.

One drawback with this procedure is that the results of our experiment will be limited in generality; that is, we could only generalize our results to people in the 100–110 range. It is exactly in this situation that the factorial experiment is ideal. In this case the two factors would be line length and IQ. More specifically, if we form a number of levels of IQ and randomly assign the subjects within these levels to the three length conditions, we will receive the benefit of a reduced error term. We will not now consider in detail how this comes about, except to say that our estimate of error variance, the within-groups mean square, is still based upon the variability of subjects treated alike and that the variability within each length-IQ condition is less than would be the case if subjects were unselected. (We will discuss this type of design in Chapter 23.)

Generality of Results

In the single-factor experiment, all variables except the one being manipulated are maintained at the same level across the different treatment groups. Such a procedure is necessary, of course, to "guarantee" that the differences observed among the treatment conditions are due solely to the operation of the independent variable. One consequence of this control is a certain lack of generality of the results; that is, the particular pattern of results may be unique to the specific values of other relevant stimulus variables maintained at a constant level throughout the course of the experiment.

The factorial experiment provides one solution to this limitation by allowing the effect of an independent variable to be averaged over several different levels of another relevant variable. As we noted in discussing the factorial arrangement in Table 9-2, the importance of line length for reading speed is assessed by comparing the scores of all of the 3-, 5-, and 7-inch subjects, one third of whom were tested at each of the three contrast levels included in the experiment. Thus, in the factorial experiment, the effect of line length represents a more general effect, averaged over three levels of contrast, than in the case of the single-factor experiment where only *one* print-paper contrast would be used. We refer to the overall effect of one independent variable, obtained by combining the scores over the different levels of the second variable, as its *main effect.* Similarly, the main effect of contrast is found by collapsing across the groups of subjects differing in lengths of line. This second main effect also represents a more general effect than would be obtained in the corresponding single-factor experiment.

The comparison between a single-factor experiment and a factorial experiment is accurate, however, only up to a point. The factorial experiment will provide the same type of information as its single-factor counterpart only when there is no *interaction* between the two independent variables. What this means is that when the effects of one of the independent variables (line length, say) are the *same* at each of the levels of the other variable (contrast) —i.e., there is *no* interaction—the main effect of line length will be the *same* as the treatment effects of line length in the single-factor experiment. On the other hand, when the effects of line length are different at the different levels of contrast (i.e., there *is* an interaction), the information provided by the main effect will *not* be the same. This is not as bad as it may sound, since the researcher will have discovered something that is *not* obtainable from the single-factor experiment, namely, the unique manner in which the two independent variables combine jointly to influence behavior. When an interaction is present, an investigator will not be interested in the main effects anyway—anything that he might say about the effects of one independent variable must be qualified by a consideration of the levels of the other. We will consider the concept of interaction next.

INTERACTION

Interaction is the one new concept that is introduced by the factorial experiment. Main effects have essentially the same meaning as in the single-factor analysis of variance and they are calculated in exactly the same way. Moreover, as we will see in later chapters, factorials with three or more variables involve no additional principles. Thus, it is important to understand the single-factor analysis of variance, since many of the principles and procedures found in this simplest of experimental designs, such as partitioning of sums of squares, the logic of hypothesis testing, and planned and post-hoc comparisons, are also found in the more complicated designs. By the same token, the two-factor analysis of variance forms a building block for designs involving three or more variables, with the concept of interaction linking them all together.

TABLE 9-3 *Example of No Interaction*

	Line Length (Factor A)			
Contrast (Factor B)	3 inches (a_1)	5 inches (a_2)	7 inches (a_3)	Mean
Low (b_1)	.89	2.22	2.89	2.00
Medium (b_2)	3.89	5.22	5.89	5.00
High (b_3)	4.22	5.55	6.22	5.33
Mean	3.00	4.33	5.00	4.11

An Example of No Interaction

One way to understand what an interaction means is to take a concrete example in which an interaction is either present or absent. Table 9-3 presents some hypothetical results for the experiment on reading speed we have been

discussing. Assume that an equal number of subjects are run in each of the nine conditions and that the values presented in the table represent the mean reading scores obtained in the experiment. The main effect of line length (factor A) is obtained by summing (or collapsing across) the three cell means for the different contrasts and then averaging these sums. The last row of the table gives these means for the three length conditions. These averages are called the column *marginal* means of the matrix. Thus, the average reading speed for subjects in the 3-inch condition is found by combining the means from the three contrast conditions and obtaining an average. In this case, we have

$$\bar{A}_1 = \frac{.89 + 3.89 + 4.22}{3} = \frac{9.00}{3} = 3.00.$$

This average reflects how fast the subjects read with 3-inch lines under three different conditions of print-paper contrast. We can obtain similar averages for the subjects reading the 5- and 7-inch materials. These are given in the other two columns.

In a like fashion, the *row* marginal averages give us information concerning the general effect of different print-paper contrasts. That is, the average reading speed for subjects in the low-contrast condition is given by an average of the means for the three length conditions. Thus,

$$\bar{B}_1 = \frac{.89 + 2.22 + 2.89}{3} = \frac{6.00}{3} = 2.00.$$

This averaging has been completed for the other contrast conditions and appears in the final column of the table.

Let's look at the two sets of marginal averages. They have been plotted in the upper two graphs of Fig. 9-1. (For the purposes of this example, we have assumed that the levels of the contrast variable are equally spaced.) In both cases we see that the independent variables influence reading scores positively, performance increasing with increases in either line length or print-paper contrast. These plots can be thought of as general descriptions of the overall effects of the two independent variables.

Now, would we say that these overall relationships are *representative* of the results obtained in the "single-factor" experiments found *within* the body of Table 9-3? There are two sets of these experiments—those reflected by the means in different rows and those reflected by the means in different columns. In the first case we are looking at the effect of varying line length (factor A) at the three different levels of contrast (factor B), while in the second case we are considering the effect of varying contrast (factor B) at the three different levels of line length (factor A). We will refer to the first set of "single-factor" experiments (the cells in the individual rows) as the *simple main effects of factor A* and to the second set (the cells in the individual columns) as the *simple main effects of factor B*. The question, then, is whether or not the simple main effects of either factor are representative of the *main* effect of the corresponding factor.

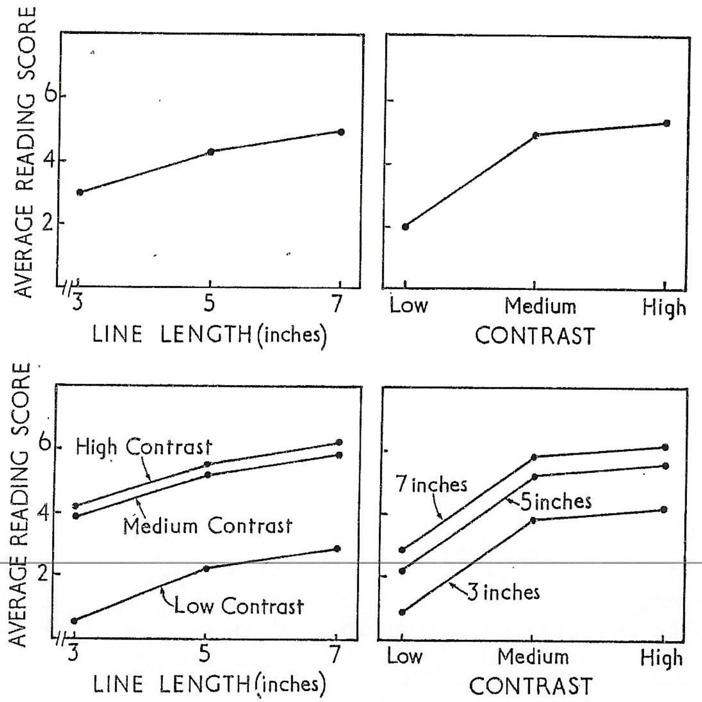

Fig. 9-1 Plot of data presented in Table 9-3; an example of no interaction.

Consider, then, the data within the body of the table. These means are presented in two double-classification plots in the lower portion of Fig. 9-1. The classification is accomplished by marking off one of the independent variables along the baseline—line length in the graph on the left—and connecting the means produced by groups receiving the same level of the other independent variable—contrast in this case. In either plotting of the results, it is clear that the *form* of the functional relationship obtained with one of the independent variables is *exactly the same* at each level of the second independent variable. The sets of functions are *parallel*, meaning that the simple main effects of either variable are the same and equal to the corresponding main effect.

An Example of Interaction

Table 9-4 presents a second set of hypothetical results using the same experimental design. Note that the same main effects are present; i.e., the means in the margins of Table 9-4 are identical to the corresponding means in Table 9-3.

There is a big difference, however, when we look at the simple main effects. To facilitate the comparisons of the simple main effects, the data within the body of the table have been plotted in Fig. 9-2. In either plot, we can see that the form of the relationship depicted by the simple main effects is *not* the same as that depicted by the row or column marginal means (the main effects). In short, then, an interaction is present.

TABLE 9-4 *Example of Interaction*

Contrast (Factor B)	Line Length (Factor A): 3 inches (a_1)	5 inches (a_2)	7 inches (a_3)	Mean
Low (b_1)	1.00	2.00	3.00	2.00
Medium (b_2)	3.00	5.00	7.00	5.00
High (b_3)	5.00	6.00	5.00	5.33
Mean	3.00	4.33	5.00	4.11

To be more specific, consider the simple main effects of line length at level b_1—the means in the first row of Table 9-4. This row is a "single-factor" experiment in which subjects from three levels of line length are tested, but *all* with a low print-paper contrast. These three means are presented in the left-hand graph of Fig. 9-2. An inspection of the figure indicates that the relationship is positive and even linear. The simple main effect at b_2 (the second row) also shows a positive linear trend for the subjects receiving the medium materials, but it is steeper than for the low-contrast case. But see what happens to the subjects tested with the high-contrast materials. In this third "single-factor" experiment the relationship is curvilinear: the reading scores first increase and then decrease with line length.

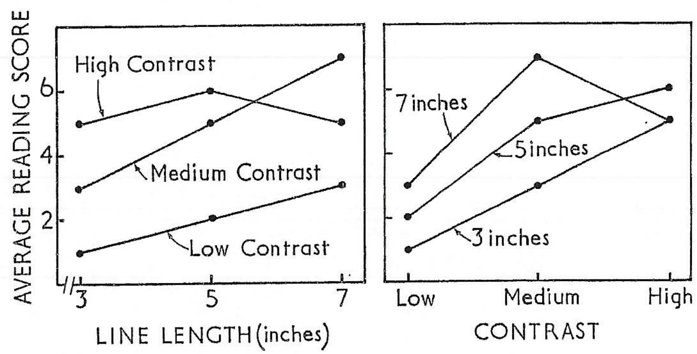

Fig. 9-2 Plot of data presented in Table 9-4; an example of interaction.

We can see the same sort of deviation of the simple main effects when we look at the means in each of the three *columns*. Here we are considering "single-factor" experiments in which contrast is varied but length is held constant. For the simple main effect of contrast for 3-inch lines (the first column) we can see that the relationship is linear. For the simple main effect for 5-inch lines (the second column) the relationship is not as sharply defined, with the function starting to "bend over" from the linear trend. In the final column (7-inch lines) we have an actual reversal of the trend—i.e., a curvilinear relationship, maximum performance being found with a medium contrast.

With either plot of the data, we can determine at a glance that the particular form of the relationship between the independent variable plotted on the baseline (line length or contrast)—i.e., the shape of the curve drawn between successive points on the baseline—is not the same at the three different levels of the other independent variable. A simple way to describe this situation is to say that the three curves are *not parallel*. When we find that an interaction is present, it is usually a good idea to plot the results of the experiment just as we have done in Fig. 9-2. The shape or form of the interaction will become readily apparent. We do not typically plot the data both ways, but choose for the baseline the independent variable that makes the most sense for the research hypotheses under consideration. Whichever way the data are plotted, an interaction will be revealed by nonparallel curves for the conditions plotted within the body of the figure.

Two Definitions of Interaction

VERBAL DEFINITION Now that we have specific examples of interaction and of lack of interaction, it is a good time to give a relatively formal definition. We say that

> **two variables interact when the effect of one variable changes at different levels of the second variable.**

An alternative way of defining interaction is to refer to the simple main effects, since a simple main effect *is* the effect of one variable at a specific level of the other variable. Thus,

> **an interaction is present when the simple main effects of one variable are not the same at different levels of the second variable.**

In the first example, the simple main effects of either variable are identical and therefore equal to the corresponding main effects. This means that row and column marginal means are perfectly representative of the effects of the two independent variables and that any subsequent analyses will generally focus upon the marginal means rather than the individual treatment means. Stated

another way, we can describe and analyze the effects of one of the independent variables without considering the specific levels of the other variable.

In the second example, the simple main effects of line length (the function relating line length and reading speed) are not the same at all levels of print-paper contrast. Stated in terms of the other variable, the simple main effects of print-paper contrast (the function relating contrast and reading speed) are not the same at all levels of line length. Either way, the data presented in Table 9-4 and plotted in Fig. 9-2 fit the definition of interaction. The presence of an interaction indicates that conclusions based on the two main effects will not fully describe the data. Each of the variables must be interpreted with the levels of the other variable in mind. To this end, any analyses conducted after the establishment of a significant interaction will tend to concentrate upon the individual treatment means rather than upon the overall marginal means.

Often the term *additive* is used to describe the joint effects of two non-interacting variables. What this means is that the effect of one variable simply adds to the effect of the second variable. When an interaction is present, the combination is *nonadditive*—i.e., an additional effect must be added to specify the joint effects of the two variables. This effect, of course, is the interaction.

ARITHMETIC DEFINITION OF INTERACTION We can translate the definition of interaction into a simple arithmetic definition and test. From the verbal statement, an interaction is present when the effects of one variable (factor A, say) change at different levels of the other variable (factor B). We have seen in Chapter 6 that it is possible to view the sum of squares for factor A in terms of differences between *pairs* of A means (pp. 85–86). It is also possible to define an interaction in terms of two of the levels of factor A rather than all a of them. That is, an interaction exists when the difference between two means at *any two* levels of factor A (a_i and $a_{i'}$) changes at *any two* levels of factor B (b_j and $b_{j'}$). (We are again referring to simple main effects, but this time defining the manipulation in terms of two levels of factor A.)

To be more specific, consider the following 2 × 2 "factorial":

	a_i	$a_{i'}$
b_j	$\overline{AB}_{ij}$	$\overline{AB}_{i'j}$
$b_{j'}$	$\overline{AB}_{ij'}$	$\overline{AB}_{i'j'}$

where the $\overline{AB}$ terms represent means obtained under the four possible treatment combinations, ab_{ij}, $ab_{i'j}$, $ab_{ij'}$, and $ab_{i'j'}$. An interaction is present if the effect of the differential A treatment (a_i versus $a_{i'}$) at one level of factor B (i.e., b_j) is not equal to the corresponding effect at another level of factor B ($b_{j'}$). In terms of simple main effects, an interaction is present when

$$A \text{ at } b_j \neq A \text{ at } b_{j'},$$

where the expression "A at b_j" represents the simple main effect of A (a_i versus $a_{i'}$ in this case) at level b_j and the expression "A at $b_{j'}$" represents the same contrast at level $b_{j'}$. Stated more quantitatively, in terms of the sets of $\overline{AB}$ means in the two rows of the table, an interaction is present when

$$\overline{AB}_{ij} - \overline{AB}_{i'j} \neq \overline{AB}_{ij'} - \overline{AB}_{i'j'}$$

or

$$(\overline{AB}_{ij} - \overline{AB}_{i'j}) - (\overline{AB}_{ij'} - \overline{AB}_{i'j'}) \neq 0. \tag{9-1a}$$

Alternatively, we may state this definition in terms of simple main effects of factor B:

$$B \text{ at } a_i \neq B \text{ at } a_{i'},$$

and in terms of the sets of means in the two columns:

$$\overline{AB}_{ij} - \overline{AB}_{ij'} \neq \overline{AB}_{i'j} - \overline{AB}_{i'j'}$$

or

$$(\overline{AB}_{ij} - \overline{AB}_{ij'}) - (\overline{AB}_{i'j} - \overline{AB}_{i'j'}) \neq 0. \tag{9-1b}$$

A number of these 2 × 2 factorial arrangements may be formed from any two-way factorial with more than two levels of either or both independent variables by simply letting the pairs of subscripts (i and i'; and j and j') take on different values. In a 3 × 3 factorial, for example, where $a = 3$ and $b = 3$, we can form 2 × 2 factorials from the crossing of a_1 and a_2 with levels b_1 and b_2, or of a_1 and a_2 with b_1 and b_3, or of a_1 and a_2 with b_2 and b_3, or of a_1 and a_3 with b_1 and b_2, and so on. Finding a nonzero value for *any one* of the possible 2 × 2 arrangements is sufficient cause to conclude that an interaction effect is present. (We would still have to assess the significance of the interaction, but for the moment we will assume that the means are free of experimental error.)

Consider the example of an interaction presented in Table 9-4. Setting $i = 1, i' = 2, j = 1$, and $j' = 2$, and substituting in Eq. (9-1a),

$$\begin{aligned}(\overline{AB}_{11} - \overline{AB}_{21}) - (\overline{AB}_{12} - \overline{AB}_{22}) &= (1.00 - 2.00) - (3.00 - 5.00)\\ &= (-1.00) - (-2.00) = 1.00.\end{aligned}$$

The nonzero value indicates that an interaction effect is present. Setting $i = 2, i' = 3, j = 1$, and $j' = 3$,

$$\begin{aligned}(\overline{AB}_{21} - \overline{AB}_{31}) - (\overline{AB}_{23} - \overline{AB}_{33}) &= (2.00 - 3.00) - (6.00 - 5.00)\\ &= (-1.00) - (1.00) = -2.00,\end{aligned}$$

also indicating an interaction. For an example of no interaction, we can look at the data in Table 9-3. Setting $i = 2, i' = 3, j = 1$, and $j' = 2$,

$$\begin{aligned}(\overline{AB}_{21} - \overline{AB}_{31}) - (\overline{AB}_{22} - \overline{AB}_{32}) &= (2.22 - 2.89) - (5.22 - 5.89)\\ &= (-.67) - (-.67) = 0;\end{aligned}$$

setting $i = 1, i' = 3, j = 1, j' = 3$,

$$\begin{aligned}(\overline{AB}_{11} - \overline{AB}_{31}) - (\overline{AB}_{13} - \overline{AB}_{33}) &= (.89 - 2.89) - (4.22 - 6.22)\\ &= (-2.00) - (-2.00) = 0.\end{aligned}$$

With this particular set of data, all possible 2 × 2 arrangements will equal zero—a necessary outcome when an interaction is absent.

We will find little need for these arithmetic tests in detecting the presence or absence of interaction in a two-factor experiment, since obviously an inspection of a double-classification plot of the data is simpler. That is, if there is no interaction, the fact that Eq. (9-1a) and Eq. (9-1b) equal zero implies that the sets of curves in the figure are *parallel*; if there is an interaction, the equations will give nonzero values which implies that the curves are *not* parallel. We will find use for the arithmetic test when we consider more complicated factorials, where the visual test is often unrevealing of interactions.

Implications of Interaction for Theory

The presence of an interaction often requires more complexity in our theoretical explanations of the data than would be the case if no interaction were present. Consider the two different outcomes we have been discussing. Both examples indicate the importance of the two independent variables. In the first case, where there is no interaction, the effect of one of the independent variables adds to the effect of the other variable. The combination is simple. In the second case, on the other hand, the combination is complex—it will take a considerable amount of theoretical ingenuity to explain why the relationship between line length and average reading score is different with the three types of print-paper contrast or why the relationship between contrast and average reading score is different for the different line lengths.

The discussion above has focussed upon the complexity of post-hoc explanations of a set of data when an interaction is found. In an increasing number of experiments being reported in the literature, interactions not only are predicted but represent the major interest of the studies. Consider, for example, research in developmental psychology. Gollin (1965) indicates that it is not particularly revealing of developmental processes simply to compare a number of different age groups on a given task. Instead, he suggests that more interesting information is obtained from the discovery of *interactions* involving some manipulated independent variables and the age dimension. To show that two age groups differ on one task but not on another allows us to speculate about different developmental processes present in the two groups and required of the two tasks. To find a main effect of age or of task suggests very little about the processes involved in the phenomena under study. As Gollin puts it, "The uncovering of both the similarity *and* the difference in performance obviously gives us an order of information about the two groups which is quite different than if we had simply demonstrated that they did or did not differ on one or the other task" (p. 166).

The discovery or the prediction of interactions may lead to a greater understanding of the behavior under study. Lashley's classic study of the effect of the amount of brain damage on maze learning by rats is an excellent example.

Lashley varied the amount of cortical tissue destroyed from a small amount (1–10 percent) to a large amount (over 50 percent) and tested these animals on three mazes differing in difficulty. He found very slight differences among the operated groups on the easiest maze, but extremely dramatic differences on the most difficult maze. If Lashley had run his animals on only one of the mazes, he would have missed this important finding: that the destruction of cortical materials affects primarily the acquisition of complex learning tasks. That is, there is no uniform *overall* learning deficit. The effect of brain damage depends upon the complexity of the material being acquired.

In short, then, if behavior is complexly determined, we will need factorial experiments to isolate and to tease out these complexities. The factorial allows us to manipulate two or more independent variables concurrently and to obtain some idea as to how the variables combine to produce the behavior. An assessment of the interaction provides a hint as to the rules of combination.

Further Examples of Interaction and Lack of Interaction

In order to broaden (and to test) your understanding of the two-variable or $A \times B$ interaction and to get some practice in extracting information from double-classification tables and plots, consider the hypothetical outcomes of a 4×3 factorial experiment in which factor A is represented at four levels and factor B at three. The means for each set of 12 treatment combinations are presented in Table 9-5.

We have seen that the means (or sums) in the margins of a two-factor matrix reflect the main effects of the two independent variables and that the means within the body of the matrix reflect the presence or absence of an interaction. In this discussion we will assume that if any differences are present among the column marginal sums or among the row marginal sums, a corresponding main effect is present, and that if the effect of one independent variable changes at the different levels of the other independent variable, an interaction is present. (As we will see, the *significance* of main effects and of interaction effects is assessed by means of an F ratio.) We will look at eight examples, representing each of the possible combinations of the presence or absence of the two main effects and the interaction effect.

Consider the first example in Table 9-5. This example represents a completely negative study, none of the three effects being present. This state of affairs is illustrated by the identical 12 means in the body of the matrix. The column marginal sums are equal, indicating the absence of a main effect of factor A; similarly, the equal row marginal sums mean an absence of a main effect of factor B. A plot of the 12 means in panel 1 of Fig. 9-3 indicates that no $A \times B$ interaction is present in the data. The second example illustrates a case in which only factor A affects performance. We may see this by inspecting the column sums, which are not equal, and the row sums, which are equal. There is also no interaction, as may be seen in panel 2 of the figure—the three curves at b_1, b_2,

TABLE 9-5 *Eight Different Outcomes of the Same Two-Factor Experiment*

Levels of Factor B	a_1	a_2	a_3	a_4	Sum	Levels of Factor B	a_1	a_2	a_3	a_4	Sum
			(1)						(2)		
b_1:	4	4	4	4	16	b_1:	2	4	6	8	20
b_2:	4	4	4	4	16	b_2:	2	4	6	8	20
b_3:	4	4	4	4	16	b_3:	2	4	6	8	20
Sum	12	12	12	12	48	Sum:	6	12	18	24	60
			(3)						(4)		
b_1:	7	7	7	7	28	b_1:	5	6	7	8	26
b_2:	6	6	6	6	24	b_2:	4	5	6	7	22
b_3:	3	3	3	3	12	b_3:	2	3	4	5	14
Sum:	16	16	16	16	64	Sum:	11	14	17	20	62
			(5)						(6)		
b_1:	1	3	5	7	16	b_1:	1	3	5	7	16
b_2:	4	4	4	4	16	b_2:	2	3.3	4.7	6	16
b_3:	7	5	3	1	16	b_3:	4	4	4	4	16
Sum:	12	12	12	12	48	Sum:	7	10.3	13.7	17	48
			(7)						(8)		
b_1:	4	5	6	7	22	b_1:	1	3	5	7	16
b_2:	4	4	4	4	16	b_2:	1	2	3	4	10
b_3:	4	3	2	1	10	b_3:	1	1	1	1	4
Sum:	12	12	12	12	48	Sum:	3	6	9	12	30

and b_3 have the identical shape. In the next example, the marginal sums show that there is a main effect of factor B but no effect of factor A. Again, no interaction is present, since the three curves at the different levels of factor B are parallel. The outcome in example 4 indicates that a main effect of both independent variables is present. This may be seen in the two sets of marginal totals. The plot in Fig. 9-3 shows that the effect of factor A is the same at each of the levels of factor B—i.e., there is no interaction.

The last four examples contain $A \times B$ interactions. Look at the marginal sums for example 5. There are no differences among the column sums and no differences among the row sums; hence, there are no main effects of factors A and B, respectively. On the basis of the main effects, then, we might conclude that our manipulations were ineffective. But look at the cell means within the body of the matrix. The two independent variables produce quite striking effects. The simple main effect of factor A is positive at b_1, absent at b_2, and negative at b_3. The $A \times B$ interaction is so severe that the simple main effects

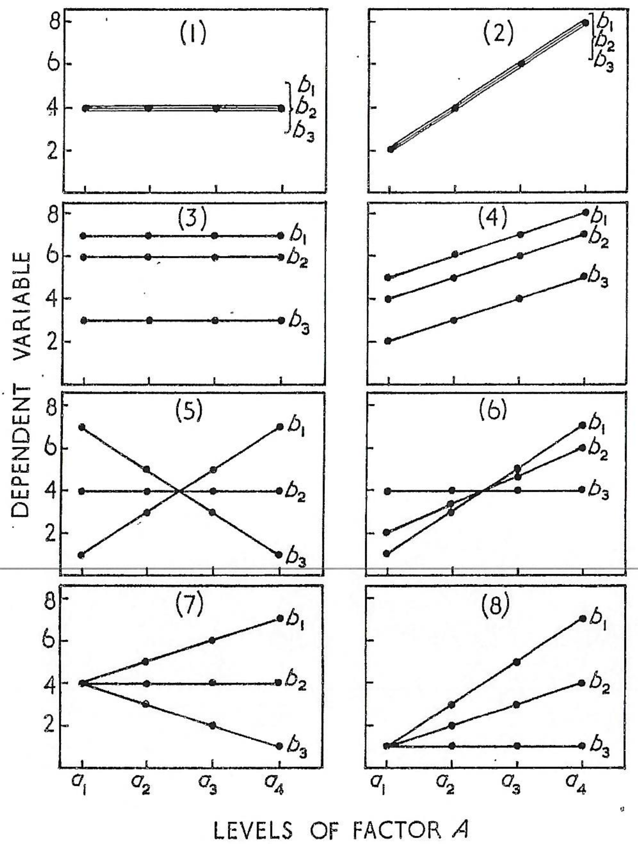

Fig. 9-3 Plot of data presented in Table 9-5.

of the two variables have been canceled. This example stresses the point that the main effects reflect treatment *averages* and as such do not necessarily reflect the constituent parts.

The next two examples show situations in which there is an interaction and one main effect. The main effect in example 6 is revealed in the column sums—i.e., a main effect of factor A—while the interaction is readily apparent in the plot of the cell means in panel 6 of Fig. 9-3. In this case, the effect of factor A is quite substantial at b_1 and nonexistent at b_3. In example 7 the situation is reversed; the row sums indicate a main effect of factor B and the nonparallel lines in panel 7 indicate an interaction of the two variables. The final experiment (example 8) provides an instance in which all three effects are present. Not only is there a main effect for each of the two variables (see the column and row

totals), but the form of the function relating factor A to the dependent variable is different at each level of factor B, indicating the presence of an $A \times B$ interaction.

We have seen that it is possible to obtain eight different combinations of the presence or absence of the two main effects and of the interaction effects. Obviously, there is an infinite number of ways in which the actual means may turn out to reflect one of these combinations. The presence of main effects is revealed by the variation among the marginal sums of the two-way matrix, and the presence of an $A \times B$ interaction is revealed by the appearance of nonparallel lines in a double-classification plot of the means within the body of the data matrix. We are now ready to see how we can obtain variances which will reflect these three effects and how we can test their significance.

chapter ten

RATIONALE AND RULES FOR CALCULATION OF THE MAJOR EFFECTS

We saw in Chapter 3 how the total sum of squares could be partitioned into two parts: (1) a part reflecting the deviation of the treatment groups from the overall mean (the between-groups sum of squares—SS_{bg}) and (2) a part reflecting the variability of subjects treated alike (the within-groups sum of squares—SS_{wg}). We then discussed how we could test the null hypothesis. In subsequent chapters of Part II we saw that we could ask more refined questions of the data by dividing the SS_{bg} into component sums of squares. The analysis of the factorial experiment follows a similar pattern, except that the SS_{bg} is *not* of systematic interest. That is, we are primarily interested in the *further division* of the SS_{bg} into three orthogonal components: (1) a sum of squares reflecting the main effect of factor A (SS_A), (2) a sum of squares reflecting the main effect of factor B (SS_B), and (3) a sum of squares representing the $A \times B$ interaction ($SS_{A \times B}$). In this chapter we will consider only the most common case, having the same number of subjects in each of the treatment conditions. The analysis of experiments with unequal sample sizes is discussed in Chapter 17.

PARTITIONING THE TOTAL SUM OF SQUARES

Design and Notation

We will pause at this point to expand the notational system so that we can make explicit the operations needed for the analysis of the two-way factorial. The system we will use is summarized in Table 10-1. The factorial arrangement of the two independent variables, illustrated with $a = 2$ and $b = 3$, is enumerated in the upper portion of the table. We have indicated that there is a total of $ab = 2(3) = 6$ treatment conditions, each with a sample of s different subjects who have been randomly assigned to the different conditions.

A basic observation or score in this design is denoted ABS_{ijk} to indicate that it represents the score of a single subject in a particular combination of the levels

TABLE 10-1 *Design and Notation for the Two-Factor Design*

EXPERIMENTAL DESIGN

	Factor A	
Factor B	a_1	a_2
b_1	$s = 4$	$s = 4$
b_2	$s = 4$	$s = 4$
b_3	$s = 4$	$s = 4$

ABS MATRIX

Treatment Combinations

ab_{11}	ab_{12}	ab_{13}	ab_{21}	ab_{22}	ab_{23}
ABS_{111}	ABS_{121}	ABS_{131}	ABS_{211}	ABS_{221}	ABS_{231}
ABS_{112}	ABS_{122}	ABS_{132}	ABS_{212}	ABS_{222}	ABS_{232}
ABS_{113}	ABS_{123}	ABS_{133}	ABS_{213}	ABS_{223}	ABS_{233}
ABS_{114}	ABS_{124}	ABS_{134}	ABS_{214}	ABS_{224}	ABS_{234}

AB MATRIX[a]

Levels of Factor B	Levels of Factor A: a_1	a_2	Marginal Sum
	Sum →		
b_1	AB_{11}	AB_{21}	B_1
b_2	AB_{12}	AB_{22}	B_2
b_3	AB_{13}	AB_{23}	B_3
Sum ↓			Sum ↓
Marginal Sum	A_1	A_2	T
	Sum →		

[a] Note: $AB_{ij} = \sum_{k}^{s} ABS_{ijk}$.

of factors A and B. These scores are arranged in the ABS matrix which appears in the middle portion of the table. If it is necessary to specify a particular score in one of the treatment conditions, we will use all three subscripts—one for the level of factor A (the i subscript), one for the level of factor B (the j subscript), and one for the score within the treatment cell (the k subscript).[1] As in the single-factor case, however, we will drop the subscripts whenever there is no ambiguity as to the arithmetic operations being specified.

An AB matrix, where the remainder of the notational system is explicated, is presented in the bottom portion of Table 10-1. The basic entry within the body of this matrix (often called the *cells* of the matrix) is the quantity AB_{ij}. This quantity represents the sum of the ABS scores at a particular combination of levels of the two factors. These are the totals that we would obtain if we summed the $s = 4$ ABS scores in any one column of the ABS matrix. For example,

$$\sum ABS_{21k} = ABS_{211} + ABS_{212} + ABS_{213} + ABS_{214} = AB_{21}.$$

More formally,

$$\sum_{k}^{s} ABS_{ijk} = AB_{ij}.$$

One way to see how the notation works is to think of the summation as canceling the relevant letter and subscript from the ABS_{ijk} designation. In this case, we are summing over the k subscript:

$$\sum_{k}^{s} ABS_{ijk} = AB\not{S}_{ij\not{k}} = AB_{ij}.$$

We will also refer to these sums as the *treatment sums* or *totals*. They form the basic ingredient in the determination of the sums of squares associated with different experimental treatments.

In order to calculate the two main effects, we will have to obtain the column and row totals shown in the margin of the matrix and hereafter referred to as the column and row marginal totals, respectively. These sums are found by collapsing across (i.e., summing over) the other subscript. More specifically, column marginal totals are formed by summing the cell totals in all of the rows for each of the a columns. These totals are denoted A_i in exactly the same manner as were the treatment sums in the single-factor case. More explicitly,

$$\sum_{j}^{b} AB_{ij} = A\not{B}_{i\not{j}} = A_i.$$

The row marginal totals are calculated in an analogous manner, summing the cell totals in all of the columns for each of the b rows. These are referred to as

[1] With these subscripts,

$$i = 1, 2, \ldots, a, \quad j = 1, 2, \ldots, b, \quad \text{and} \quad k = 1, 2, \ldots, s.$$

B_j. That is,

$$\sum_i^a AB_{ij} = AB_{ij} = B_j.$$

Finally, the grand total is obtained by summing either set of marginal totals:

$$T = \sum A = \sum B.$$

Component Deviation Scores

Suppose factor A was manipulated at $a = 3$ levels and factor B at $b = 2$ levels, so that we had a total of $ab = 3(2) = 6$ different treatment groups, each containing s subjects. As a first step, it is useful to think of the six treatment means as coming from a single-factor experiment. According to the formulas given in Chapter 3, the variability of the *abs* subjects can then be broken down into

$$SS_T = SS_{bg} + SS_{wg}.$$

Up to this point, then, there is nothing new to the analysis. We will now refine the SS_{bg}.

The SS_{bg} is based upon the deviation of each individual treatment mean from the total mean—i.e., $\overline{AB}_{ij} - \overline{T}$. Consider the deviation score produced by a group of subjects receiving a particular treatment combination represented by the combination of level a_i and level b_j (ab_{ij}). This deviation score can be influenced by three sources of variability:

$$\overline{AB}_{ij} - \overline{T} = (A_i \text{ effect}) + (B_j \text{ effect}) + (A_i \times B_j \text{ interaction effect}).$$

Each of these effects can be expressed as a deviation score involving familiar quantities:

$$\overline{AB}_{ij} - \overline{T} = (\overline{A}_i - \overline{T}) + (\overline{B}_j - \overline{T}) + (\overline{AB}_{ij} - \overline{A}_i - \overline{B}_j + \overline{T}). \qquad (10\text{-}1)$$

Suppose we try to understand Eq. (10-1) a little better. First, we can verify that the equation is correct by performing the indicated additions and subtractions. To be more specific, there is only one $\overline{AB}_{ij}$ on the righthand side of Eq. (10-1) and so it will stay, but $\overline{A}_i$ and $\overline{B}_j$ will both drop out, since each appears once as a positive quantity and once as a negative quantity. The final term, $\overline{T}$, appears three times on the right—twice as a negative quantity and once as a positive quantity. Thus, we are left with the same expression, $\overline{AB}_{ij} - \overline{T}$, on both sides of the equation.

The second point concerns the specification of the interaction effect. To show that the third quantity on the right of Eq. (10-1) reflects an interaction, we can redefine an interaction as a *residual* deviation score. That is, the interaction effect represents whatever is left of the deviation of the individual treatment

mean from $\overline{T}$ that cannot be accounted for by the two relevant main effects. In symbols,

$$\begin{aligned}\text{interaction effect} &= (\text{deviation from } \overline{T}) - (A_i \text{ effect}) - (B_j \text{ effect}) \\ &= (\overline{AB}_{ij} - \overline{T}) - (\overline{A}_i - \overline{T}) - (\overline{B}_j - \overline{T}).\end{aligned}$$

Performing some simple algebra, we obtain

$$\begin{aligned}\text{interaction effect} &= \overline{AB}_{ij} - \overline{T} - \overline{A}_i + \overline{T} - \overline{B}_j + \overline{T} \\ &= \overline{AB}_{ij} - \overline{A}_i - \overline{B}_j + \overline{T}.\end{aligned}$$

We are now ready to consider the deviation scores for the individual subjects in the different treatment groups. We can easily expand Eq. (10-1) to accommodate the deviation of any given subject (ABS_{ijk}) from the mean of all of the subjects ($\overline{T}$). A complete subdivision of the total deviation ($ABS_{ijk} - \overline{T}$) is given by

$$\begin{aligned}ABS_{ijk} - \overline{T} = (\overline{A}_i - \overline{T}) + (\overline{B}_j - \overline{T}) + (\overline{AB}_{ij} - \overline{A}_i - \overline{B}_j + \overline{T}) \\ + (ABS_{ijk} - \overline{AB}_{ij}). \qquad (10\text{-}2)\end{aligned}$$

In words, the deviation of a subject from the grand mean can be broken down into four separate components: (1) the treatment effect at level a_i, (2) the treatment effect at level b_j, (3) the interaction effect at the combination of levels a_i and b_j, and (4) the deviation of the subject from his individual treatment mean.

Now that we have enumerated the component deviation scores for each subject, they can be squared and summed to produce the corresponding sums of squares for the analysis. However, rather than looking at the actual defining formulas, which preserve the "meaning" of Eq. (10-2), we will move directly to the computational formulas. These formulas are much easier to use than the defining formulas. Thus, except for the explication of the component deviation scores [i.e., Eq. (10-2)], no real purpose is served by looking at the defining formulas for the different sums of squares. We will now turn to a specification of these computational formulas.

RULES FOR GENERATING COMPUTATIONAL FORMULAS

In this section we will consider a set of rules which allow the generation of the computational formulas for the sums of squares obtained from *any* factorial experiment. The system will be introduced here in the context of the two-way factorial, but (as we will see in later chapters) it can be applied to a large variety of experimental designs. The method we will discuss, a modified version of the one presented by Myers (1966), is based on an isomorphic relationship between the *df* statement for a given source of variance and the corresponding formula for the sum of squares. The main purpose of this section is to introduce this useful scheme for constructing the computational formulas for the different component sums of squares. In the next section we will summarize the complete analysis and discuss in more detail the meaning behind some of the operations.

The system consists of three basic steps. First, we identify the sources of variance which are extracted in a standard analysis of variance. Second, we write the *df* statement for each of these sources. Finally, we construct the computational formulas from the different *df* statements.

Identification of Sources of Variance

There is a simple rule for specifying the sources of variance. We have already discussed what these sources would be in the present case, but it is useful to see how the rule applies in a situation with which we are familiar. This rule "works" with completely randomized designs of the sort we are discussing in the present major section and in Part IV.

(1) List all factors, including the within-groups factor, and (2) Form all possible interactions with these factors, omitting the within-groups factor.

For the two-factor design, step 1 results in a listing of

$$A, B, \text{ and } S/AB.$$

(The within-groups factor, S/AB, represents the variability due to subjects treated alike; that is, this source consists of the variability of subjects in each of the *ab* groups, pooled or summed over these different groups.) Step 2 results in the listing of a single interaction:

$$A \times B.$$

Degrees of Freedom

We will discuss the meaning of degrees of freedom in the next section. For the present we will just consider formulas that specify the *df*'s for the different sources of variance we have identified. For the two main effects, the *df*'s are simply the number of levels for each factor minus 1:

$$df_A = a - 1 \quad \text{and} \quad df_B = b - 1.$$

For the $A \times B$ interaction, the *df* are the product of the *df*'s associated with factors A and B. More specifically,

$$df_{A \times B} = (df_A)(df_B) = (a - 1)(b - 1). \qquad (10\text{-}3)$$

The calculation of the *df* for the within-groups source, S/AB, is more complicated. The variability for this source is due to a subject factor (factor S), and for this factor

$$df_S = s - 1.$$

However, since this factor is present in each of the *ab* treatment conditions, the *df* for S/AB is found by multiplying df_S times the total number of these groups, *ab*:

$$df_{S/AB} = (df_S)(ab) = (s - 1)(ab) = ab(s - 1).$$

The last expression on the right represents the *df* statement in the most common form. Finally, the *df* for the total sum of squares consist of the total number of observations (*abs*) minus 1:

$$df_T = abs - 1.$$

Construction of Computational Formulas

Table 10-2 summarizes the steps followed in the generation of the computational formulas for the different sums of squares in a two-way analysis of variance. The sources of variance are listed in the first column; the *df*'s associated with these sources are listed in the second column. We will outline the construction procedure as a series of steps.

FIRST STEP **Write the expanded *df* statement for the source of variance.** This first step represents the backbone of the computational scheme. It should present no difficulty, since it requires only the performance of the multiplications specified in the *df* statements in column 2 of Table 10-2. After completing any necessary multiplication, we arrange the sets of letters according to decreasing numbers of letters. When present, "1" is listed last. The expanded *df* statements are given in column 3 of the table.

Each term in these expanded *df* statements—single letters, combination of letters, or "1"—denotes a different term in the computational formulas, and the expanded *df* statements themselves indicate how these terms are to be combined to produce the different sums of squares. We will now construct the computational expressions for each of these quantities.

Look over the expanded *df* statements. Except for "1," each term appears once in the first position to the left. Thus, we will construct the computational expressions for these terms in the rows where they appear in the first position. (The last row of the table lists the term associated with "1" in the expanded *df* statements.)

SECOND STEP For each letter in the first terms, **substitute capital letters; for "1" substitute "*T*."** This has been done in column 4 of the table. These letters and combinations of letters denote sums in particular matrices. (We have omitted subscripts, as they will not be necessary for the computational formulas.) That is, A refers to the overall total for any level of factor A, B to the overall total for any level of factor B, AB to the total for any cell in the AB matrix, ABS to the score for a single subject in the ABS matrix, and T to the sum of all of the scores in the experiment.

THIRD STEP Next, we **square and sum all such totals or scores.** This step is enumerated in column 5 of the table. Consistent with the conventions adopted for this book, single summation signs without limits and quantities without subscripts are used, since the summation is taken over *all* such quantities. That is, the summation in the first row includes all a of the squared totals, in the second row all b of the squared totals, in the third row all ab of the squared totals, and in the fourth and fifth rows all abs of the squared scores.

FOURTH STEP In this step we **select the appropriate divisor** for each term. We find these numbers easily by starting with the total number of observations in the experiment (abs) and *eliminating* those letters that appear in each numerator. This step is completed in column 6.

FIFTH STEP The final operation on these first terms is to **represent each term by a code letter.** The principle behind the code is obvious, as a comparison of column 6 and column 7 will reveal. That is, the letter(s) within brackets correspond to the letter(s) used to represent the quantities which are squared in the numerator of the coded term.

SIXTH STEP The **computational formulas are written in coded form** in the last column of the table. The particular combination of terms for each sum of squares is specified by the corresponding df statement in column 3. It is interesting to note the correspondence between these coded formulas and the respective deviation scores in Eq. (10-2).

Summary

This system is general and may be applied to all of the designs we will consider in this book. In addition, we will see in Chapter 15 that the formulas for other sorts of analyses can be generated by the same underlying system. The system elaborated here ensures that we will never "forget" the computational formulas, since we can very easily *reconstruct* them. Some of the steps will drop out with practice, and (as we will see in Chapter 15) shorthand forms can be used to simplify the formulas still further. You should not lose touch with the basic system, however, as it will prove extremely useful in generating formulas for the more complex designs considered in Parts IV and V.

In the next section we will consider the computational formulas again, but this time in conjunction with the remaining steps in the analysis of variance.

TABLE 10-2 *Construction of the Computational Formulas from the Expanded df Statements*

			First Term				
(1) Source	(2) df	(3) Expanded df	(4) Substitution	(5) Square and Sum	(6) Selection of Denominator	(7) Letter Code	(8) Computational Formula (Coded)[a]
A	$a-1$	$a-1$	A	$\sum (A)^2$	$\frac{\sum (A)^2}{abs} = \frac{\sum (A)^2}{bs}$	$[A]$	$[A]-[T]$
B	$b-1$	$b-1$	B	$\sum (B)^2$	$\frac{\sum (B)^2}{abs} = \frac{\sum (B)^2}{as}$	$[B]$	$[B]-[T]$
$A \times B$	$(a-1)(b-1)$	$ab-a-b+1$	AB	$\sum (AB)^2$	$\frac{\sum (AB)^2}{abs} = \frac{\sum (AB)^2}{s}$	$[AB]$	$[AB]-[A]-[B]+[T]$
S/AB	$ab(s-1)$	$abs-ab$	ABS	$\sum (ABS)^2$	$\frac{\sum (ABS)^2}{abs} = \sum (ABS)^2$	$[ABS]$	$[ABS]-[AB]$
Total	$abs-1$	$abs-1$	ABS	$\sum (ABS)^2$	$\frac{\sum (ABS)^2}{abs} = \sum (ABS)^2$	$[ABS]$	$[ABS]-[T]$
	1	1	T	$(T)^2$	$\frac{(T)^2}{abs} = \frac{(T)^2}{abs}$	$[T]$	

[a] Bracketed letters represent complete terms in the computational formulas; a particular term is identified by the letter(s) appearing in the numerator.

SUMMARY OF THE ANALYSIS OF VARIANCE

Sums of Squares

The computational formulas for the component sums of squares are presented in Table 10-3. Each term in the computational formulas is expressed in its complete form only once—when it first appears in the analysis. Thereafter, each term is designated by the letter code in which a particular term is identified by the letter or letters appearing in the numerator. The totals required for the SS_A are the column marginal totals in the AB matrix presented in Table 10-1 (p. 188). The totals required for the SS_B come from the row marginal totals in the AB matrix. The totals for the first term in the computational formula for the $SS_{A \times B}$ are the individual cell totals found within the body of the AB matrix. Finally, the scores for the first term in the formula for the $SS_{S/AB}$ appear in the ABS matrix.

The within-groups sum of squares ($SS_{S/AB}$) reflects the variability of subjects treated alike. That is, it consists of the variability of subjects receiving the same treatment combination, pooled over all of the ab treatment groups. While the correspondence between the computational formula and the deviation score ($ABS_{ijk} - \overline{AB}_{ij}$) is apparent, the fact that this sum of squares represents a pooling of the separate within-group sums of squares for the different ab groups is not.[2]

It is informative to see how computational rule number 1 (pp. 44–45) applies to the different formulas in Table 10-3. This rule states that we always divide a numerator term by the number of observations summed to calculate

TABLE 10-3 *Computational Formulas: Two-Factor Analysis of Variance*

Source	Computational Formula[a]	df	MS	F[b]
A	$\frac{\sum (A)^2}{bs} - \frac{(T)^2}{abs}$	$a - 1$	$\frac{SS_A}{df_A}$	$\frac{MS_A}{MS_{S/AB}}$
B	$\frac{\sum (B)^2}{as} - [T]$	$b - 1$	$\frac{SS_B}{df_B}$	$\frac{MS_B}{MS_{S/AB}}$
$A \times B$	$\frac{\sum (AB)^2}{s} - [A] - [B] + [T]$	$(a - 1)(b - 1)$	$\frac{SS_{A \times B}}{df_{A \times B}}$	$\frac{MS_{A \times B}}{MS_{S/AB}}$
Within (S/AB)	$\sum (ABS)^2 - [AB]$	$ab(s - 1)$	$\frac{SS_{S/AB}}{df_{S/AB}}$	
Total	$\sum (ABS)^2 - [T]$	$abs - 1$		

[a] Bracketed letters represent complete terms in the computational formulas; a particular term is identified by the letter(s) appearing in the numerator.
[b] The *fixed-effects model* is assumed (see footnote 3, p. 200).

[2] This fact is demonstrated in Exercise 2 for this part (p. 247).

one of the quantities that is squared. For the first term in the formula for the SS_A, we square sums that have been obtained by collapsing across the different levels of factor B. With s subjects and b groups, there are bs observations contributing to any one of the marginal sums (A_i). Thus, the rule applies, and we divide the numerator by the number of observations represented by each marginal sum (bs). The same is true for the second term of the computational formula, where the squared quantity (T) is the grand sum of all of the abs observations in the experiment. For the first term for the SS_B, the squared totals are the marginal sums B_j, which are obtained by summing across the different levels of factor A; there are as observations when this is done. The rule holds again. For the first term for the $SS_{A \times B}$ we are asked to square AB sums, each of which contains s observations, and this is the number by which we divide the numerator. For the $SS_{S/AB}$ and the SS_T the squared term (ABS) is based on a single observation, and so we can think of the quantity $\sum (ABS)^2$ as being divided by 1. In each case, then, computational rule number 1 continues to hold.

Degrees of Freedom

The df for any source of variance satisfy the statement given in Eq. (4-2): the df equal the number of independent observations upon which each sum of squares is based minus the number of restraints operating on these observations. For the two main effects, the observations involved are the marginal sums (or means) for the rows and columns. For the SS_A the number of observations is a; since the column marginal totals (A_i) must sum to the grand total (T), there is one restraint. Thus,

$$df_A = a - 1.$$

This restraint is symbolized in Table 10-4 by an **X** placed in the margin at a_4. (This level was arbitrarily picked; any of the a levels would do.) For the SS_B the number of observations is b; since the marginal row totals (B_j) must also sum to T, one restraint is placed on the independence of these observations. Thus,

$$df_B = b - 1.$$

This restraint is symbolized by an **X** placed in the margin at b_3.

The df for the $SS_{A \times B}$ are obtained from Eq. (10-3):

$$df_{A \times B} = (a - 1)(b - 1),$$

the product of the df associated with the two main effects. We may understand this formula by considering the cells within the AB matrix presented in Table 10-4. The question basically is how many of the AB totals are *free* to vary once certain restrictions of the matrix are met. We have already seen that the marginal sums for the columns and rows must satisfy the requirement that

$$\sum A = \sum B = T.$$